# Finiteness of moments for zeros of Gaussian fields

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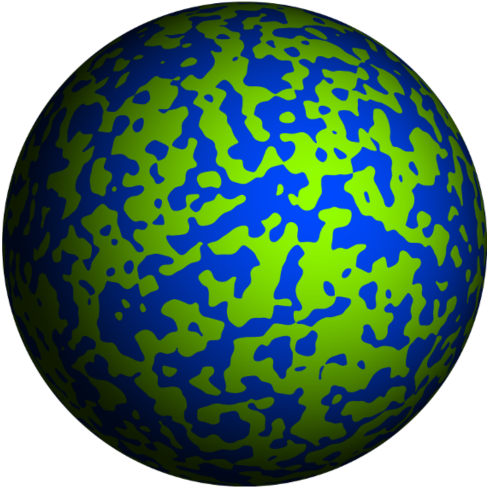
Mathematical Imaging and Random Geometry in Nice
November 25th, 2024

# Random hypersurfaces



A random algebraic curve of degree 1000 on $\mathbb{S}^2$. Credits: Vincent Beffara.

# Gaussian vectors

Let $V$ be Euclidean space with an orthonormal basis $(e_1, \ldots, e_n)$.

A random vector $N = \sum_{i=1}^{n} N_i e_i \in V$ is *centered* if $\mathbb{E}[N_i] = 0$ for all $i$.
In this case $\mathrm{Var}(N) = \left(\mathbb{E}[N_i N_j]\right)_{1 \leqslant i,j \leqslant n} \in \mathcal{S}_n^{\geqslant}(\mathbb{R})$.

For $\Lambda \in \mathcal{S}_n^{\geqslant}(\mathbb{R})$, denote by $\mathcal{N}(\Lambda)$ the *centered Gaussian of variance* $\Lambda$.

- $N \sim \mathcal{N}(\Lambda)$ in $V$ is *non-degenerate* if $\Lambda > 0$.
- If $L : V \to W$ is linear then $LN \sim \mathcal{N}(L\Lambda L^*)$ in $W$.
- If $N$ is non-degenerate then: $LN$ non-degenerate $\iff$ $L$ surjective.

# Gaussian fields

A *Gaussian field* is a random function $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that:
$\forall p \in \mathbb{N}^*$, $\forall x_1, \ldots, x_p \in \mathbb{R}^n$, $(f(x_1), \ldots, f(x_p)) \in \mathbb{R}^p$ is a centered Gaussian.

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Denote $X = (X_1, \ldots, X_n)$ and for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$:

$$\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}, \quad |\alpha| = \sum \alpha_i, \quad \alpha! = \prod \alpha_i! \quad \text{and} \quad X^\alpha = \prod X_i^{\alpha_i}.$$

$f$ is *p-non-degenerate* if: $\forall x \in \mathbb{R}^n$, $(\partial^\alpha f(x))_{|\alpha| \leqslant p}$ is non-degenerate.

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$f$ is *$p$-non-degenerate* if: $\forall x \in \mathbb{R}^n$, $(\partial^\alpha f(x))_{|\alpha| \leqslant p}$ is non-degenerate.

- Kac Polynomials: $\sum_{|\alpha| \leqslant d} N_\alpha X^\alpha$, where $(N_\alpha)$ independent $\mathcal{N}(1)$.
- Bargmann–Fock field: $\sum_{\alpha \in \mathbb{N}^n} N_\alpha \frac{X^\alpha}{\sqrt{\alpha!}}$, where $(N_\alpha)$ independent $\mathcal{N}(1)$.
- Berry field: stationary Gaussian field such that $-\Delta f = f$ almost surely.

# Nodal volume

Let $f$ be a non-degenerate Gaussian field and $Z = f^{-1}(0) \subset \mathbb{R}^n$.

**Theorem (Lerario–Stecconi, 2019)**

*Almost surely $Z$ is a smooth boundaryless hypersurface (possibly empty).*

Interested in the $(n-1)$-dimensional $\mathrm{Vol}(Z \cap B)$, where $B$ is the unit ball.

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**Some previous finiteness results for $\mathbb{E}[\mathrm{Vol}(Z \cap B)^p]$**

- Bulinskaya (1961): $n = 1$, $p = 1$ (stationary $\mathcal{C}^1$ fields).
- Cramer–Leadbetter (1967): $n = 1$, $p = 2$ (stationary $\mathcal{C}^2$ fields).
- Nualart–Wschebor (1991): $n = 1$, $p > 2$ ($\mathcal{C}^{2p+2}$ fields).
- Adler (1981): $n > 1$, $p = 1$ ($\mathcal{C}^1$ fields).

# Main result

**Theorem (Ancona–L., 2023)**

*If $f$ is $(p-1)$-non-degenerate then $\mathbb{E}\left[\operatorname{Vol}(Z \cap B)^p\right] < +\infty$.*

*More generally, let $F : M \to \mathbb{R}^k$ be $\mathcal{C}^p$, Gaussian, $(p-1)$-non-degenerate, then for any compact $K \subset M$ we have $\mathbb{E}\left[\operatorname{Vol}\left(F^{-1}(0) \cap K\right)^p\right] < +\infty$.*

- Independent proof by Gass–Stecconi (2023).
- $f : \mathbb{R}^n \to \mathbb{R}$, proof by Armentano–Azaïs–Ginsbourger–Leòn (2019). Their proof does not adapt to our setting.

# Kac–Rice formula for the expectation

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a non-degenerate Gaussian field and $Z = f^{-1}(0)$.

## Kac–Rice formula

For any open $U \subset \mathbb{R}^n$:

$$\mathbb{E}\left[\mathrm{Vol}(Z \cap U)\right] = \int_U \underbrace{\frac{\mathbb{E}[\|\nabla_x f\| \mid f(x) = 0]}{\sqrt{2\pi \,\mathrm{Var}(f(x))}}}_{=\rho_1(x)} \, \mathrm{d}x.$$

- $\rho_1(x)$ only depends on $\mathrm{Var}(f(x), \nabla_x f)$.
- $\rho_1 : \mathbb{R}^n \to [0, +\infty)$ is continuous.

Same kind of formula for a non-degenerate Gaussian field $F : M \to \mathbb{R}^k$.

# Kac–Rice formula for moments

Denote $\underline{x} = (x_1, \ldots, x_p) \in (\mathbb{R}^n)^p$ and $\Delta = \{\underline{x} \in (\mathbb{R}^n)^p \mid \exists i \neq j, x_i = x_j\}$.

## Kac–Rice formula for the $p$-th moment

If $(f(x_1), \ldots, f(x_p))$ is non-degenerate for all $\underline{x} \in (\mathbb{R}^n)^p \setminus \Delta$ then:

$$\mathbb{E}\left[\operatorname{Vol}(Z \cap B)^p\right] = \int_{B^p} \underbrace{\frac{\mathbb{E}\left[\prod_{i=1}^p \|\nabla_{x_i} f\| \;\middle|\; \forall i,\ f(x_i) = 0\right]}{\det\big(2\pi \operatorname{Var}(f(x_1), \ldots, f(x_p))\big)^{\frac{1}{2}}}}_{=\rho_p(\underline{x})} \, \mathrm{d}\underline{x}.$$

$\rho_p : (\mathbb{R}^n)^p \to [0, +\infty)$ is continuous on $(\mathbb{R}^n)^p \setminus \Delta$ but singular along $\Delta$.

## Goal

Prove that $\rho_p$ is locally integrable on $(\mathbb{R}^n)^p$.

# Changing the field

**Sketch of proof of the Kac–Rice formula for the $p$-th moment.**

Since $\Delta$ is negligible in $Z^p$:

$$\big(\mathrm{Vol}_{n-1}(Z\cap B)\big)^p = \mathrm{Vol}_{p(n-1)}(Z^p\cap B^p) = \mathrm{Vol}_{p(n-1)}\big((Z^p\setminus\Delta)\cap B^p\big).$$

Apply the formula for $p=1$ to $F:\underline{x}\mapsto(f(x_1),\dots,f(x_p))$ on $B^p\setminus\Delta$. □

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If $G : (\mathbb{R}^n)^p \setminus \Delta \to V$ is non-degenerate Gaussian and $G^{-1}(0) \overset{\text{a.s.}}{=} Z^p \setminus \Delta$, the same proof gives an alternative expression of $\rho_p$.

**Example**

$G(\underline{x}) = L(\underline{x})F(\underline{x})$ where $L : (\mathbb{R}^n)^p \setminus \Delta \to GL_p(\mathbb{R})$ is smooth.

If $G$ extends non-degenerate over $\Delta$, then $\rho_p$ extends $\mathcal{C}^0$ over $\Delta$.

# An example for $n = 1$ and $p = 2$

For $x \neq y \in \mathbb{R}$ define

$$G(x,y) = \begin{pmatrix} 1 & 0 \\ -\frac{1}{y-x} & \frac{1}{y-x} \end{pmatrix} \begin{pmatrix} f(x) \\ f(y) \end{pmatrix} = \begin{pmatrix} f(x) \\ \frac{f(y)-f(x)}{y-x} \end{pmatrix} \xrightarrow[y \to x]{} \begin{pmatrix} f(x) \\ f'(x) \end{pmatrix}.$$

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## New goal

Find $G : (\mathbb{R}^n)^p \setminus \Delta \to V$ such that $G^{-1}(0) \overset{\text{a.s.}}{=} Z^p \setminus \Delta$ which extends over $\Delta$ into a non-degenerate Gaussian field.

For $\underline{x} \in (\mathbb{R}^n)^p$, denote $\mathrm{ev}_{\underline{x}} : f \mapsto (f(x_1), \dots, f(x_p))$.

$$F(\underline{x}) = \underbrace{\mathrm{ev}_{\underline{x}}}_{\substack{\text{geometric,} \\ \text{degenerated when } \underline{x} \in \Delta}} \underbrace{(f)}_{\substack{\text{random,} \\ \text{nice}}}.$$

# Case $n = 1$, $p \geqslant 1$: Hermite interpolation

**Hermite polynomial of $f$ at $\underline{x} = (\underbrace{y_1, \ldots, y_1}_{k_1+1}, \ldots, \underbrace{y_m, \ldots, y_m}_{k_m+1}) \in \mathbb{R}^p$**

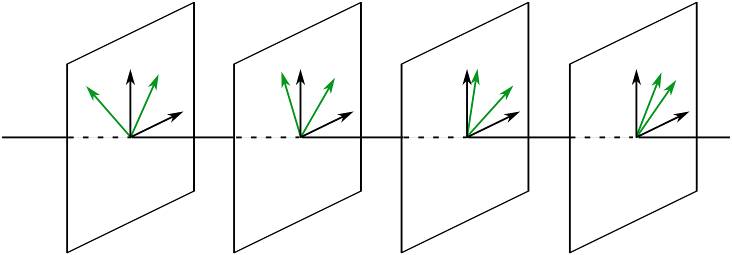
There is a unique $P \in \mathbb{R}_{p-1}[X]$ such that $\left(P^{(k)}(y_i)\right)_{\substack{i \leqslant m \\ k \leqslant k_i}} = \left(f^{(k)}(y_i)\right)_{\substack{i \leqslant m \\ k \leqslant k_i}}$.
Denote it by $P = K(f, \underline{x})$.

- $K(\cdot, \underline{x}) : \mathcal{C}^\infty(\mathbb{R}) \to \mathbb{R}_{p-1}[X]$ is linear surjective for all $\underline{x} \in \mathbb{R}^p$.
- $K(f, \cdot) : \mathbb{R}^p \to \mathbb{R}_{p-1}[X]$ is smooth for any $f \in \mathcal{C}^\infty(\mathbb{R})$.
- If $\underline{x} \notin \Delta$, then $K(f, \underline{x}) = 0 \iff \left(f(x_1), \ldots, f(x_p)\right) = 0 \iff \underline{x} \in Z^p$.

If $f$ is $(p-1)$-non-degenerate then $G = K(f, \cdot) : \mathbb{R}^p \to \mathbb{R}_{p-1}[X]$ works.

# A vector bundle perspective

Instead of $K(f, \cdot)$ consider $\underline{x} \mapsto \big(\underline{x}, K(f, \underline{x})\big)$ and a basis depending on $\underline{x}$.



Sketch of a moving frame. Credits: Cécile Clavaud.

- Lagrange ($\underline{x} \notin \Delta$): $L_i(\underline{x}) = \prod_{j \neq i} \frac{X - x_j}{x_i - x_j}$ and $K(f, \underline{x}) = \sum f(x_i) L_i(\underline{x})$.
- Newton: $N_i(\underline{x}) = \prod_{j<i} (X - x_i)$ and $K(f, \underline{x}) = \sum f[x_1, \ldots, x_i] N_i(\underline{x})$.

  For $p = 2$, $K(f, x, y) = f(x) + \frac{f(y) - f(x)}{y - x}(X - x)$.

# Case $p = 2$, $n \geqslant 1$: Polar coordinates

For $x \neq y \in \mathbb{R}^n$,

$$G(x,y) = \left(f(x), \frac{f(y)-f(x)}{\|y-x\|}\right) = 0 \iff f(x) = 0 = f(y).$$

## Polar change of variable

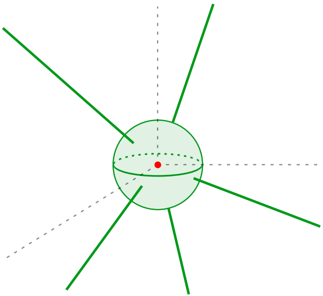
$$\widetilde{G}(x,R,\theta) = G(x, x+R\theta) \xrightarrow[R\to 0]{} (f(x), D_x f(\theta)).$$

$G$ does not extend to $(\mathbb{R}^n)^2$ but $\widetilde{G}$ extends to $\mathbb{R}^n \times [0,+\infty) \times \mathbb{S}^{n-1}$.

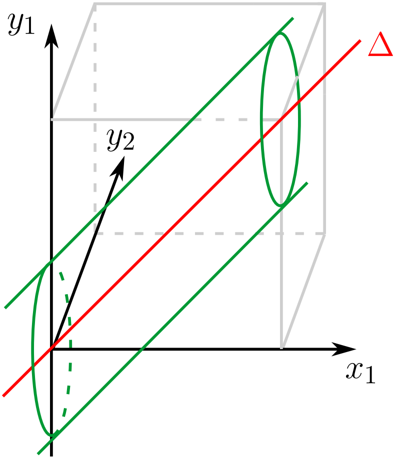
Enough to prove $\rho_2$ locally integrable on $(\mathbb{R}^n)^2$.

Geometrically, $G$ extends to $\mathrm{Bl}_\Delta\left((\mathbb{R}^n)^2\right) \simeq \left((\mathbb{R}^n)^2 \setminus \Delta\right) \sqcup \Delta \times \mathbb{S}^{n-1}$.

# Blow-ups, or how to make polar coordinates complicated



(a) Blow-up of 0 in $\mathbb{R}^3$.




(b) 3D slice of the blow-up of $\Delta$ in $(\mathbb{R}^2)^2$.

Credits: Cécile Clavaud.

# Mid-talk summary

## New goal

Find $G : (\mathbb{R}^n)^p \setminus \Delta \to V$ such that $G^{-1}(0) \overset{\text{a.s.}}{=} Z^p \setminus \Delta$ which extends over $\Delta$ into a non-degenerate Gaussian field.

## Strategy

1. Consider zeros with multiplicity and interpolate by polynomials.
2. Extend $G$ not to $(\mathbb{R}^n)^p$ but to some larger space, geometric counterpart of a polar change of variable.

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## Strategy

1. Consider zeros with multiplicity and interpolate by polynomials.

   Pb: if $n \geqslant 2$, no Hermite interpolation in $\mathbb{R}^n$.

2. Extend $G$ not to $(\mathbb{R}^n)^p$ but to some larger space, geometric counterpart of a polar change of variable.

   Pb: if $p \geqslant 3$, no polar coordinates for $p$ points nor nice $\mathrm{Bl}_\Delta((\mathbb{R}^n)^p)$.

# Divided differences

$$\sigma_{p-1} = \left\{ \underline{t} = (t_1, \dots, t_p) \in [0,1]^p \;\middle|\; \sum t_i = 1 \right\}$$

$\mathrm{d}\underline{t}$ Lebesgue measure such that $\int_{\sigma_{p-1}} \mathrm{d}\underline{t} = \frac{1}{(p-1)!}$.

$\sigma_{p-1}$ for $1 \leqslant p \leqslant 4$.

## Divided differences

For all $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $\underline{x} = (x_1, \dots, x_p) \in (\mathbb{R}^n)^p$, define

$$f[\underline{x}] = \int_{\sigma_{p-1}} D^{(p-1)}_{\sum t_i x_i} f \,\mathrm{d}\underline{t} \qquad \in \mathrm{Sym}^{p-1}(\mathbb{R}^n).$$

- $f[x_1] = f(x_1)$,
- $f[x_1, x_2] = \int_0^1 D_{x_1 + s(x_2 - x_1)} f \,\mathrm{d}s$,
- $f[\underbrace{x, \dots, x}_{p \text{ times}}] = \frac{1}{(p-1)!} D^{(p-1)}_x f$.

# Kergin interpolation

**Kergin polynomial of $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ at $\underline{x} = (x_1, \ldots, x_p) \in (\mathbb{R}^n)^p$**

There exists a unique $P \in \mathbb{R}_{p-1}[X]$ such that $f[(x_i)_{i \in I}] = P[(x_i)_{i \in I}]$ for all non-empty $I \subset \{1, \ldots, p\}$. Denote it by $P = K(f, \underline{x})$.

**Example**

$K(f, x, \ldots, x)$ is the Taylor polynomial of order $p-1$ of $f$ at $x$.

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**Example**

$K(f, x, \dots, x)$ is the Taylor polynomial of order $p-1$ of $f$ at $x$.

- $K(\cdot, \underline{x})$ is linear surjective for all $\underline{x} \in (\mathbb{R}^n)^p$.
- $K(f, \cdot)$ is smooth for any $f \in \mathcal{C}^\infty(\mathbb{R}^n)$.
- $K(f, \underline{x})$ interpolates $f$ at $x_i$ with multiplicity. Given $\underline{x} \notin \Delta$,

$$\underbrace{K(f, \underline{x}) = 0}_{\binom{n+p}{n} \text{ conditions}} \implies \underbrace{(f(x_1), \dots, f(x_p)) = 0}_{p \text{ conditions}} \quad \text{but not the converse.}$$

# Kernel of $\mathrm{ev}_{\underline{x}} : f \mapsto \big(f(x_1), \ldots, f(x_p)\big)$

If $x \notin \Delta$, then $\mathrm{ev}_{\underline{x}} : \mathbb{R}_{p-1}[X] \to \mathbb{R}^p$ is surjective. Denote $\Gamma(\underline{x}) = \ker(\mathrm{ev}_{\underline{x}})$.

$$\mathrm{ev}_{\underline{x}}(f) = 0 \iff \mathrm{ev}_{\underline{x}}(K(f, \underline{x})) = 0 \iff \underbrace{K(f, \underline{x}) \bmod \Gamma(\underline{x})}_{=\mathrm{mj}(f, \underline{x})} = 0.$$

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$$\begin{array}{c} \mathcal{C}^\infty(\mathbb{R}^n) \\ {\scriptstyle K(\cdot,\underline{x})}\downarrow \\ \mathbb{R}_{p-1}[X] \\ {\scriptstyle \mathrm{ev}_{\underline{x}}}\downarrow \\ \mathbb{R}^p \end{array}$$

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\ker(\mathrm{ev}_{\underline{x}}) = \Gamma(\underline{x}) & \hookrightarrow & \mathbb{R}_{p-1}[X] & \twoheadrightarrow & \mathbb{R}_{p-1}[X]/\Gamma(\underline{x}) \\
 & & \Big\downarrow{\scriptstyle \mathrm{ev}_{\underline{x}}} & \swarrow{\scriptstyle \simeq} & \\
 & & \mathbb{R}^p & &
\end{array}$$

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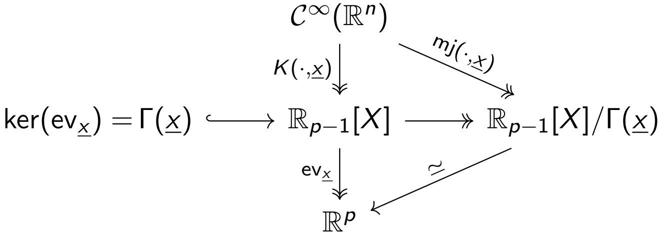


$\Gamma : (\mathbb{R}^n)^p \setminus \Delta \to \mathrm{Gr}_p(\mathbb{R}_{p-1}[X])$ smooth, hence $\mathrm{mj}(f,\cdot)$ works on $(\mathbb{R}^n)^p \setminus \Delta$.

Need to extend $\Gamma$ smoothly over $\Delta$.

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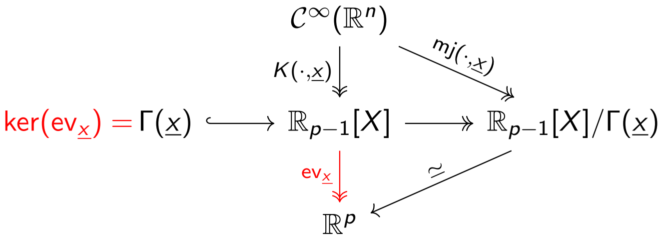


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# An example for $n = 2$ and $p = 2$

$\mathrm{Gr}_2(\mathbb{R}_1[X])$ set of lines in $\mathbb{R}_1[X_1, X_2] = \mathrm{Span}(1, X_1, X_2)$.

If $x = R\big(\cos(\theta), \sin(\theta)\big) \neq 0$, then $\Gamma(0, x) = \mathrm{Span}\big(\sin(\theta)X_1 - \cos(\theta)X_2\big)$.

- $\Gamma(0, x)$ encodes the line $(0, x) \subset \mathbb{R}^2$.
- $P$ mod $\Gamma(0, x)$ can be thought of as the restriction of $P$ to $(0, x)$.
- $\Gamma(0, \cdot)$ extends smoothly to $\mathrm{Bl}_0(\mathbb{R}^2)$ but not to $\mathbb{R}^2$.

# Compactification of configuration spaces

**Theorem (Ancona–L., 2023)**

*There exists a smooth manifold without boundary $C_p[\mathbb{R}^n]$ of dimension $np$ and a smooth proper surjection $\pi : C_p[\mathbb{R}^n] \to (\mathbb{R}^n)^p$ such that:*

- $\pi$ *diffeomorphism from a dense open subset in* $C_p[\mathbb{R}^n]$ *to* $(\mathbb{R}^n)^p \setminus \Delta$.
- $\Gamma$ *extends smoothly to* $C_p[\mathbb{R}^n]$.

For $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $z \in C_p[\mathbb{R}^n]$ define $\mathrm{mj}(f, z) = K(f, \pi(z)) \bmod \Gamma(z)$.

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- *$\pi$ diffeomorphism from a dense open subset in $C_p[\mathbb{R}^n]$ to $(\mathbb{R}^n)^p \setminus \Delta$.*
- *$\Gamma$ extends smoothly to $C_p[\mathbb{R}^n]$.*

For $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $z \in C_p[\mathbb{R}^n]$ define $\mathrm{mj}(f,z) = K(f,\pi(z)) \bmod \Gamma(z)$.

If $f$ is $(p-1)$-non-degenerate then $\mathrm{mj}(f,\cdot)$ is non-degenerate and

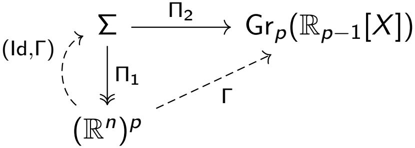
$$\forall \underline{x} \in (\mathbb{R}^n)^p \setminus \Delta, \qquad \big(f(x_1),\dots,f(x_p)\big) = 0 \iff \mathrm{mj}(f,\underline{x}) = 0.$$

# Resolution of singularities

$\Gamma : (\mathbb{R}^n)^p \setminus \Delta \to \mathrm{Gr}_p(\mathbb{R}_{p-1}[X])$ is algebraic, hence

$$\Sigma = \overline{\left\{(\underline{x}, \Gamma(\underline{x})) \;\middle|\; \underline{x} \in (\mathbb{R}^n)^p \setminus \Delta\right\}} \subset (\mathbb{R}^n)^p \times \mathrm{Gr}_p(\mathbb{R}_{p-1}[X])$$

is an algebraic manifold, a priori singular.



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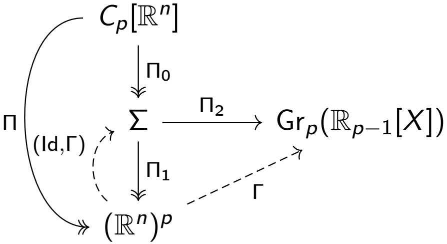
**Theorem (Hironaka, 1964)**

*$\Sigma$ admits a resolution of singularities $C_p[\mathbb{R}^n]$.*



# The end

Thank you for your attention.